PRiME 2017 Summary: Algebraic Geometry

Edray Goins

April 3, 2018

Edray Goins was the research mentor of a group of five undergraduates (Chineze Egbunike Christopher (Purdue University), Robert Julian Dicks (Emory University), Gina Marie Ferolito (Wellesley College), Joseph M. Sauder (Pontifical Catholic University of Puerto Rico), and Danika Keala Van Niel (Syracuse University)) as well as two graduate research assistants (Jacob Bond (Purdue University) and Abhishek Parab (Purdue University)).

Research Project

Background

Gennadiĭ Belyĭ proved in 1979 that a compact connected Riemann surface S of genus g is completely determined by the existence of a rational map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has three critical values. We say that a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ is a rational map with critical values $\{0,\,1,\,\infty\}$; and that a Dessin d'Enfant (French for "child's drawing") is a bipartite graph with "black" vertices $\beta^{-1}(0)$, "white" vertices $\beta^{-1}(1)$, midpoints of faces $\beta^{-1}(\infty)$, and edges $\beta^{-1}\bigl([0,1]\bigr)$. Alexander Grothendieck wrote about this result in 1984: "This discovery, which is technically so simple, made a very strong impression on me [...]. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper [...] gives a perfectly explicit example."

Given a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ with $\deg(\beta) = N$, denote $\widetilde{X} = \bigl\{P \in S \bigm| \beta(P) \neq 0,\,1,\,\infty\bigr\}$ as a Riemann surface of genus g and $X = \bigl\{x_0 \in \mathbb{P}^1(\mathbb{C}) \bigm| x_0 \neq 0,\,1,\,\infty\bigr\}$ as the thrice punctured sphere. For each a point $x_0 \in X$, the fundamental group is a free group on two generators:

$$\pi_1\bigl(X,\,x_0\bigr) = \bigl\{\gamma : [0,1] \to X \bigm| \gamma(0) = \gamma(1) = x_0\bigr\} \big/ \sim = \bigl\langle \gamma_0,\,\gamma_1,\,\gamma_\infty \bigm| \gamma_0 \circ \gamma_1 \circ \gamma_\infty = 1 \bigr\rangle .$$

Consider the preimage $V = \beta^{-1}(x_0) = \bigl\{P_1,\,P_2,\,\ldots,\,P_N\bigr\}$ as a subset of $\widetilde{X}$. For every closed loop $\gamma \in \pi_1\bigl(X,\,x_0\bigr)$, there is a path $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ such that $\beta \circ \widetilde{\gamma} = \gamma$. In fact, $\widetilde{\gamma}(0) = P_i$ while $\widetilde{\gamma}(1) = P_j$ for some $P_i,\,P_j \in V$, so we find a permutation $\sigma \in S_N$ on the N subscripts such that $j = \sigma(i)$. That is, there is a group homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(V) \to S_N$ which sends a closed loop γ to a permutation σ, so let $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ be those permutations which are the images of the generators $\gamma_0,\,\gamma_1,\,\gamma_\infty \in \pi_1(X, x_0)$.

In 1891, Adolf Hurwitz showed the following four non-trivial properties.

i. The composition $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ is the trivial permutation, and the subgroup $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle$ of the symmetric group S_N generated by them is a transitive subgroup. This is called the monodromy group of β.

ii. Each of these permutations is a product of disjoint cycles:

$$\begin{aligned} \sigma_0 &= \prod_{P \in B} \left(b_{P,1}\ b_{P,2}\ \cdots\ b_{P,e_P}\right) & & B = \beta^{-1}(0) \\ \sigma_1 &= \prod_{P \in W} \left(w_{P,1}\ w_{P,2}\ \cdots\ w_{P,e_P}\right) & \text{corresponding to} \qquad & W = \beta^{-1}(1) \\ \sigma_\infty &= \prod_{P \in F} \left(f_{P,1}\ f_{P,2}\ \cdots\ f_{P,e_P}\right) & & F = \beta^{-1}(\infty) \end{aligned}$$

where $e_P = \#\left\{Q \in S \,|\, \beta(Q) = \beta(P)\right\}$ is the ramification index of $P \in S$.

iii. The multiset $\mathcal{D} = \bigl\{\{e_P \bigm| P \in B\},\ \{e_P \bigm| P \in W\},\ \{e_P \bigm| P \in F\}\bigr\}$ is a collection of positive integers such that $N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + (2\,g - 2)$.

iv. Conversely, any multiset $\mathcal{D}$ which a collection of three multisets is the degree sequence for some Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ if and only if there exist permutations $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ such that the first three properties above hold.

There seems to be growing interest in these topics throughout the STEM fields. For example, Belyĭ maps $\beta : S \to \mathbb{P}^1(\mathbb{C})$ are of interest to physicists due to the applications in String Theory, while monodromy groups $\mathrm{Mon}(\beta) = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ are of interest to algebraic geometers and number theorists alike due to their relation with the Inverse Galois Problem.

Goals

The PRiME 2015 project sought (i) to compile and to create several examples of Belyĭ pairs and (ii) to create plots of their corresponding Dessins d'Enfant. The PRiME 2016 project sought (i) to start a database of Belyĭ pairs and (ii) to compute their monodromy groups. The PRiME 2017 project sought (i) to complete the database of Belyĭ pairs and their monodromy groups, (ii) to compute monodromy groups of compositions of Belyĭ maps, and (iii) to focus on monodromy groups of those Dessins d'Enfant which are toroidal graphs.

Project #1: Towards a Database of Belyĭ Maps

We began PRiME 2017 by following ideas in a 2013 paper by Michael Klug, Michael Musty, Sam Schiavone, and John Voight. They explained how to start with a degree N to compute all possible Belyĭ maps $\beta : S \to \mathbb{P}^1(\mathbb{C})$ having $\deg(\beta) = N$ for a compact connected Riemann surface S of genus g. Their algorithm consists of the following steps:

#1. List all $\mathcal{D} = \big\{ \{e_P \,|\, P \in B\},\ \{e_P \,\big|\, P \in W\},\ \{e_P \,\big|\, P \in F\} \big\}$ such that $N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + (2\,g - 2)$ for some nonnegative integer g.

#2. For each such degree sequence $\mathcal{D}$, group the integers 1, 2, ..., N to form permutations σ_0', σ_1', $\sigma_\infty' \in S_N$ as disjoint products of cycles of "cycle type $\mathcal{D}$" as in Hurwitz's theorem.

#3. Compute the double coset decomposition $H \backslash G / K = \bigcup_i \overline{g_i}$ for the symmetric group $G = S_N$ as well as the centralizers $H = C_G(\sigma_0')$ and $K = C_G(\sigma_1')$. (This step is quite time intensive.)

#4. Keep only those double coset representatives $g \in S_N$ such that the permutations $\sigma_0 = \sigma_0$, $\sigma_1 = g \circ \sigma_1' \circ g^{-1}$ and $\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$ satisfy the properties (i) the permutations σ_0, σ_1, $\sigma_\infty \in S_N$ are of "cycle type $\mathcal{D}$" and (ii) the group $\big\langle \sigma_0, \sigma_1, \sigma_\infty \big\rangle$ is a transitive subgroup of S_N.

Before the summer began, my graduate student Jacob Bond had computed the number of such Belyĭ pairs (S, β) for $N \leq 11$; see Table 1.

Degree N	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$N = 1$	1	0	0	0	0	0
$N = 2$	1	0	0	0	0	0
$N = 3$	2	1	0	0	0	0
$N = 4$	6	2	0	0	0	0
$N = 5$	14	9	4	0	0	0
$N = 6$	63	70	16	0	0	0
$N = 7$	269	443	182	30	0	0
$N = 8$	1336	3255	2245	385	0	0
$N = 9$	6988	23971	23895	7450	900	0
$N = 10$	39304	177247	256041	131928	19344	0
$N = 11$	224640	1326642	2660722	1996108	516100	54990

Table 1: The Number of of Belyĭ maps $\beta : S \to \mathbb{P}^1(\mathbb{C})$ having $\deg(\beta) = N$ where S has genus g

Since it is possible to draw a Dessin d'Enfant directly from permutations σ_0, σ_1, $\sigma_\infty \in S_N$ satisfying $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$, I asked the students in PRiME 2017 (i) to determine the structure of monodromy groups

$\langle\sigma_0,\,\sigma_1,\,\sigma_\infty\rangle \leq S_N$ and (ii) to draw the Dessins d'Enfant for the permutations computed for $N \leq 6$. You can see some examples for $N = 5$ and $g = 1$ in Figure 1.

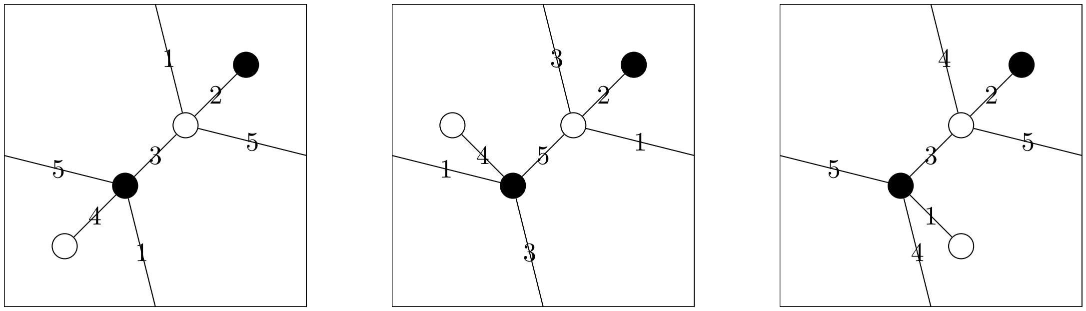

Figure 1: Three Toroidal Graphs Corresponding to $\mathcal{D} = \big\{\{1,\,4\},\,\{1,\,4\},\,\{5\}\big\}$

For instance, the monodromy group $\langle\sigma_0,\,\sigma_1,\,\sigma_\infty\rangle$ for the toroidal graph on the left can be found by reading off the edge labeling tracing counter-clockwise: $\sigma_0 = (1\,3\,5\,4)\;(2)$, $\sigma_1 = (1\,3\,5\,2)\;(4)$, and $\sigma_\infty = (1\,4\,3\,2\,5)$. All of this has been written up for a forthcoming paper. (The paper is still being written; it has not been submitted yet.)

Project #2: Determining Monodromy Groups of Belyĭ Maps

For PRiME 2017, we decided to view a compact connected Riemann surface S as the collection of complex-valued projective points $(x : y : 1) \in \mathbb{P}^2(\mathbb{C})$ which are the set of zeroes of some polynomial $f(x, y)$ with complex coefficients, and we view a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ as a ratio $\beta(x, y) = p(x, y)/q(x, y)$ of two polynomials with complex coefficients. For example, we can view the Riemann sphere $S = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$ as the zero locus of $f(x, y) = y$ via the embedding $S \hookrightarrow \mathbb{P}^2(\mathbb{C})$ which sends $(x : y : 1) \mapsto (x : 0 : 1)$.

I tasked the students in PRiME 2017 with writing a computer program which would compute the transitive subgroup $\mathrm{Mon}(\beta) = \langle\sigma_0,\,\sigma_1,\,\sigma_\infty\rangle$ of a given Belyĭ pair (S, β) with $\deg(\beta) = N$. They were to write code to generate $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ for three polynomials $f(x, y)$, $p(x, y)$ and $q(x, y)$ by implementing the following algorithm:

#1. Numerically compute the preimage

$$V = \beta^{-1}(x_0) = \left\{ P \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; f(P) = p(P) - x_0\, q(P) = 0 \right\} = \big\{P_1,\,P_2,\,\ldots,\,P_N\big\}$$

for $x_0 = 1/2$ as a subset of the Riemann surface $\widetilde{X} = \big\{P \in S \;\big|\; \beta(P) \neq 0,\,1,\,\infty\big\}$.

#2. Choose $\varepsilon = 0,\,1$. As we range over all points $P_i \in V$, use Euler's method to numerically compute the unique solution $\widetilde{\gamma}_\varepsilon^{(i)} : [0, 1] \to \mathbb{P}^2(\mathbb{C})$ to the initial value problem

$$\left.\begin{aligned} \frac{d\widetilde{\gamma}_\varepsilon^{(i)}}{dt} &= F_\varepsilon\big(\widetilde{\gamma}_\varepsilon^{(i)}\big) \\ \widetilde{\gamma}_\varepsilon^{(i)}(0) &= P_i \end{aligned}\right\} \quad \text{where} \quad F_\varepsilon = \frac{2\,\pi\,\sqrt{-1}\,(p - \varepsilon\, q)\, q}{q\left(\dfrac{\partial f}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial p}{\partial x}\right) - p\left(\dfrac{\partial f}{\partial x}\dfrac{\partial q}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial q}{\partial x}\right)} \begin{bmatrix} -\dfrac{\partial f}{\partial y} \\ +\dfrac{\partial f}{\partial x} \end{bmatrix}.$$

(These are chosen so that $\beta \circ \widetilde{\gamma}_\varepsilon^{(i)} = \gamma_\varepsilon$ in terms of the loop $\gamma_\varepsilon(t) = \varepsilon + (x_0 - \varepsilon)\, e^{2\pi\sqrt{-1}t}$ around ε satisfying $\gamma_\varepsilon(0) = \gamma_\varepsilon(1) = x_0$.)

#3. Compute $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ such that (i) $\widetilde{\gamma}_\varepsilon^{(i)}(1) = P_{\sigma_\varepsilon(i)}$ for $\varepsilon = 0,\,1$ and $i = 1,\,2,\,\ldots,\,N$ and (ii) $\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$.

The group worked tirelessly to write code in `Sage` which computed not only the permutations $\sigma_0, \sigma_1, \sigma_\infty \in S_N$ from a given Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ but to also call `Pari/GP` to determine the structure of the group $\mathrm{Mon}(\beta) = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$. We attempted to do this in PRiME 2016, but we failed. The students in PRiME 2017 found a way to bypass all of the problems from that summer and to write code which worked beautifully!

Project #3: Regular Toroidal Graphs

For PRiME 2017, we knew that triples of permutations $\sigma_0, \sigma_1, \sigma_\infty \in S_N$ satisfying $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ corresponded to Dessins d'Enfant, so we wanted to see how much we could say about the monodromy groups $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ just from the cycle types alone. To this end, we focused on regular toroidal graphs; that is, regular bipartite graphs which can be embedded on the torus $\mathbb{T}^2(\mathbb{R})$ without edge crossings but not on the sphere $S^2(\mathbb{R})$. Some examples of regular toroidal graphs can be found in Figure 2.

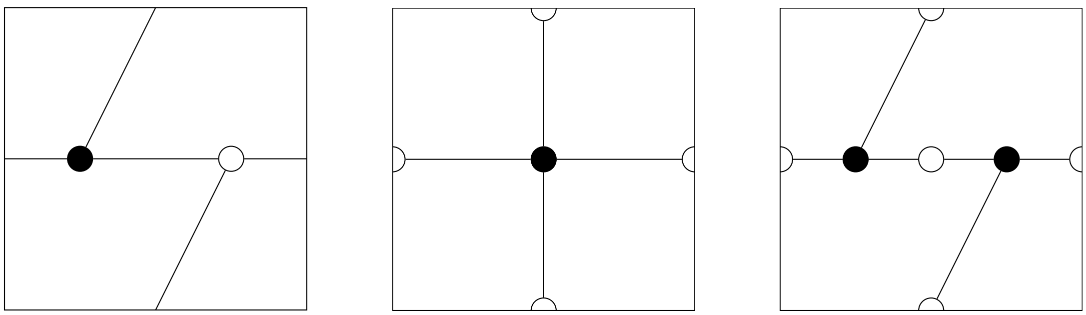

Figure 2: Toroidal Graphs for $\big\{\{3\}, \{3\}, \{3\}\big\}$, $\big\{\{4\}, \{2, 2\}, \{4\}\big\}$, and $\big\{\{3, 3\}, \{2, 2, 2\}, \{6\}\big\}$

It is easy to see that there are only three possible degree sequences; see Table 2. The students observed right away that there can be many non-isomorphic regular toroidal graphs corresponding to the same degree sequence by drawing the graphs as in Figure 3. The students found several other examples as well, and subsequently worked to determine the number of non-isomorphic regular toroidal graphs for a given degree sequence $\mathcal{D}$ as above. While they had some rather clever ideas, they were unable to come up with a specific formula before the summer ended.

Degree Sequence $\mathcal{D}$	$\lvert B\rvert$	$\lvert W\rvert$	$\lvert F\rvert$	Degree N
$\Big\{\underbrace{\{3, 3, \dots, 3\}}_{n \text{ times}}, \underbrace{\{3, 3, \dots, 3\}}_{n \text{ times}}, \underbrace{\{3, 3, \dots, 3\}}_{n \text{ times}}\Big\}$	n	n	n	$3\,n$
$\Big\{\underbrace{\{2, 2, \dots, 2\}}_{2\,n \text{ times}}, \underbrace{\{4, 4, \dots, 4\}}_{n \text{ times}}, \underbrace{\{4, 4, \dots, 4\}}_{n \text{ times}}\Big\}$	$2\,n$	n	n	$4\,n$
$\Big\{\underbrace{\{2, 2, \dots, 2\}}_{3\,n \text{ times}}, \underbrace{\{3, 3, \dots, 3\}}_{2\,n \text{ times}}, \underbrace{\{6, 6, \dots, 6\}}_{n \text{ times}}\Big\}$	$3\,n$	$2\,n$	n	$6\,n$

Table 2: Degree Sequences for the Regular Toroidal Graphs

Nonetheless, by focusing on several examples, we have found that the monodromy groups (in the literature, they are actually called cartographic groups) are "usually" the semi-direct products of cyclic groups

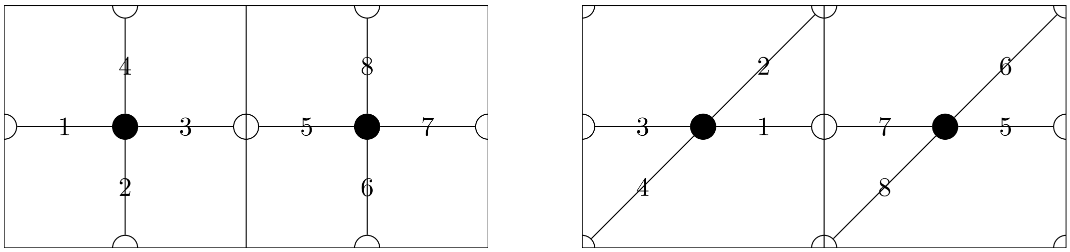

Figure 3: Two Regular Toroidal Graphs Corresponding to $\mathcal{D} = \big\{\{4, 4\}, \{2, 2, 2, 2\}, \{4, 4\}\big\}$

as $\big(C_n \times C_n\big) \rtimes C_3$, $\big(C_n \times C_n\big) \rtimes C_4$, or $\big(C_n \times C_n\big) \rtimes C_6$. We were able to prove that these are always the monodromy groups when the number of faces n is divisible only by primes p in the form $p \equiv 1 \bmod 3$, $p \equiv 1 \bmod 4$, or $p \equiv 1 \bmod 6$, respectively. We found counter examples for other types of n. Unfortunately, we still do not have a clean classification to explain what happens for general n.

Other PRiME Activities

Welcome Orientation

On Sunday June 4, 2017, we had a Welcome Orientation for the PRiME participants. In the evening, after the students checked into the Hilltop Apartments at Purdue University, the housing staff gave a short orientation to the campus, and then we ended the evening by enjoying pizza and introductions. (See Figure 4.)

Figure 4: Welcome Orientation on June 4, 2017

CAARMS23

From Wednesday June 21 through Saturday June 24, 2017, PRiME send the Algebraic Geometry group to the 23rd annual Conference for African American Researchers in the Mathematical Sciences (CAARMS). The conference was held at the University of Michigan in Ann Arbor. Five undergraduate students (Chineze Christopher, Robert Dicks, Gina Ferolito, Joseph Sauder, and Danika Van Niel) drove from West Lafayette to Ann Arbor for the four-day conference. The students each presented a poster on Thursday evening, then attended a banquet on Friday evening. Robert Dicks won for Best Undergraduate Poster, and subsequently gave a presentation on his research that Saturday morning. (See Figure 5.) Edray Goins wrote a blog post for the American Mathematical Society on the experience:

`https://blogs.ams.org/inclusionexclusion/2017/07/15/caarms23/`

Figure 5: CAARMS23 from June 21–24, 2017

Guest Speakers

PRiME invited several professors at Purdue to give presentations to the students. For example, on Thursday, June 29, 2017, Mark Ward (Professor of Statistics) addressed the group on Generating Functions. (See Figure 6.)

Graduate School Panel

On Thursday, July 20, 2017, PRiME ran a panel at Purdue University to discuss graduate programs in the mathematical sciences. The panel featured the Director of Graduate Studies, David Goldberg, as well as graduate students from the Department of Mathematics (Chris Creighton, Alejandra Gaitan, Alison Rosenblum, and Kelsey Walters) and the Department of Statistics (Tim Keaton and Yumin Zhang). The ten students from PRiME asked the panel various questions for over an hour. (See Figure 7.)

Indiana Undergraduate Research Conference

On Tuesday, July 25, 2017, Purdue University hosted the annual Indiana Undergraduate Mathematics Conference. The conference featured morning parallel sessions, a break for lunch at the Tudor Room in the

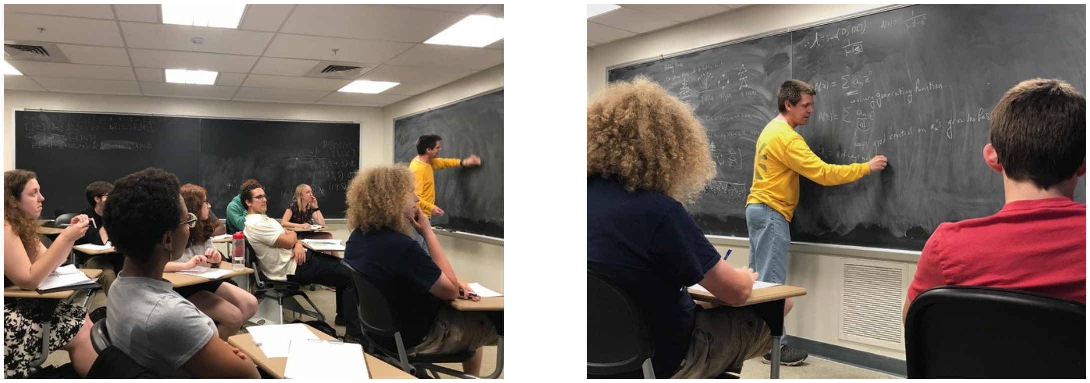

Figure 6: Mark Ward on June 29, 2017

Figure 7: Graduate School Panel on July 20, 2017

Purdue Memorial Union including chats with current undergraduate students conducting research all over Indiana, afternoon parallel sessions, and a concluding plenary speaker highlighting mathematics and business/industry. The conference brought in 60 students (26 groups gave talks) and faculty from six campuses (Indiana University, IUPUI, Purdue, Rose-Hulman, Valparaiso University, and Wabash College) throughout the state. (See Figure 8

Final Presentations

On Monday, July 24, 2017, each of the two PRiME groups presented their final results at Purdue University. Each group spent one hour giving a talk for students and faculty at Purdue. (See Figure 9.)

MAA MathFest

From Wednesday July 26 through Saturday July 29, 2017, PRiME attended the MAA MathFest. The conference was held in Chicago. All ten undergraduate students drove to Chicago for the final three days of the conference. Each of the Algebraic Geometry and Probability groups gave talks at the conference. (See Figure 10.)

Figure 8: Indiana Undergraduate Research Conference on July 25, 2017

Deliverables

The students did an excellent job at presenting at various conferences.

- They each presented a separate poster at the 23rd annual Conference for African American Researchers in the Mathematical Sciences (CAARMS) which was held from June 21–24, 2017 at the University of Michigan.
- Robert Dicks won a prize for Best Undergraduate Poster at CAARMS23, and gave a talk on Saturday June 24.
- Edray Goins wrote a blog post for the American Mathematical Society on the group's experience at CAARMS23.
- The group gave a 20-minute talk and presented a poster at the 2017 Indiana Undergraduate Mathematics Conference on July 25 at Purdue University.
- The group gave a 20-minute talk at the Mathematical Association of America (MAA) MathFest which was held from July 26–29, 2017 in Chicago, IL.

As mentioned before, there is a preprint containing write-ups of the results discovered over the summer. It is being prepared for submission to a journal.

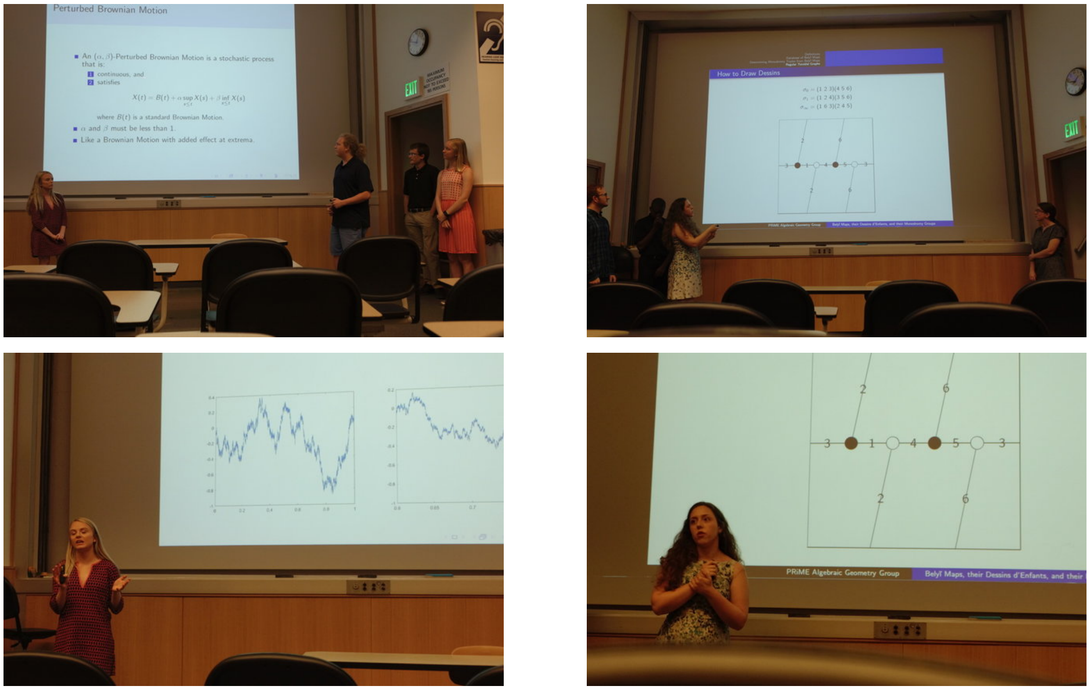

Figure 9: Final Presentations on July 24, 2017

Figure 10: MAA MathFest from July 26–29, 2017